Quasi-arithmetic lattices and their hyperbolic volumes (Topology and Analysis of Discrete Groups and Hyperbolic Spaces)

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Quasi-arithmetic lattices and their hyperbolic volumes

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1 Hyperbolic manifolds and lattices

Let $\mathbf{H}^n$ be the hyperbolic n-space, that is, the complete simply connected Riemannian n-space of constant curvature -1. In the projective model of $\mathbf{H}^n$ the group of isometries of $\mathbf{H}^n$ corresponds to the Lie group

$$\mathrm{PO}(n,1) = \mathrm{O}(n,1)/\{\pm 1\},$$

where $\mathrm{O}(n,1)$ denotes the orthogonal group of the standard Lorentzian quadratic form of signature $(n,1)$.

A discrete subgroup $\Gamma \subset \mathrm{PO}(n,1)$ is called a (hyperbolic) *lattice* if the quotient space $\Gamma\backslash\mathbf{H}^n$ has finite volume. We call a lattice $\Gamma \subset \mathrm{PO}(n,1)$ *uniform* (or *cocompact*) if $\Gamma\backslash\mathbf{H}^n$ is compact.

Remark 1.1. More generally a lattice in a locally compact group G is a discrete subgroup $\Gamma \subset G$ such that $\Gamma\backslash G$ possesses a finite right G-invariant measure. In case $G = \mathrm{PO}(n,1)$ this definition coincides with the above (one uses the fact that the isotropy groups for the action of $\mathrm{PO}(n,1)$ on $\mathbf{H}^n$ are compact).

In case the discrete subgroup $\Gamma \subset \mathrm{PO}(n,1)$ is torsion-free we have that $M = \Gamma\backslash\mathbf{H}^n$ is a hyperbolic manifold, that is, this quotient is locally isometric to $\mathbf{H}^n$. Since $\mathbf{H}^n$ is simply connected, we have that $\pi_1(M) \cong \Gamma$. If Γ contains nontrivial torsion element, then $\Gamma\backslash\mathbf{H}^n$ has singularities. In this general case we speak of hyperbolic *orbifold* for this quotient.

2 Arithmetic subgroups

A powerful way to construct hyperbolic (as well as other) lattices is to consider the so-called arithmetic subgroups. We recall their construction here. We start with a linear algebraic k-group $\mathbf{G}$, for some number field k. The ring of integers $\mathcal{O}_k$ in k is naturally embedded as a discrete subgroup of $k \otimes_{\mathbb{Q}} \mathbb{R}$. More precisely, it is a (Euclidean) lattice in the latter. After fixing a matrix representation $\mathbf{G} \subset \mathrm{GL}_N$ defined over $\mathbb{Q}$ we can consider the group of integer points:

$$\mathbf{G}(\mathcal{O}_k) = \mathbf{G}(k) \cap \mathrm{GL}_N(\mathcal{O}_k). \tag{2.1}$$

This group (contrarily to the k-points) depends on the choice of the embedding $\mathbf{G} \subset \mathrm{GL}_N$, but its commensurability class in $\mathbf{G}(k)$ is completely determined by the k-group $\mathbf{G}$. Moreover, it is clear that $\mathbf{G}(\mathcal{O}_k)$ is discrete in $\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R})$. Note that "being discrete" is a property invariant by commensurability. The same is true for "being a lattice" or "being uniform". And in many cases $\mathbf{G}(\mathcal{O}_k)$ is a indeed a lattice:

Theorem 2.1 (Borel and Harish-Chandra). *If $\mathbf{G}$ is semisimple then $\mathbf{G}(\mathcal{O}_k)$ is a lattice in $\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R})$.*

To construct a lattice in $\mathrm{PO}(n,1)$ we can now proceed as follows. Let k be a number field with a fixed embedding $k \subset \mathbb{R}$, and $\mathbf{G}$ be an algebraic k-group such that $\mathbf{G}(\mathbb{R}) = \mathrm{PO}(n,1)$ (this forces $\mathbf{G}$ to be semisimple). This implies that $\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R}) = \mathrm{PO}(n,1) \times K$ for some factor K, with the projection onto the first factor being induced by the map $k \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{R}$ given by $1 \otimes x \mapsto x$. Then $\mathbf{G}(\mathcal{O}_k)$ is a lattice in $\mathrm{PO}(n,1) \times K$. Under the additional assumption that K is compact, after projection $\mathbf{G}(\mathcal{O}_k)$ becomes a lattice in $\mathrm{PO}(n,1)$.

Remark 2.2. The condition "K compact" forces the field k to be totally real.

Remark 2.3. By passing to a finite extension of k we can assume that $\mathbf{G}$ is absolutely simple in this construction.

In this context we define:

Definition 2.4. An absolutely simple k-group $\mathbf{G}$ with $\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R}) \cong \mathrm{PO}(n,1) \times K$ for K compact is called *admissible* (for $\mathrm{PO}(n,1)$).

Definition 2.5. Any subgroup $\Gamma \subset \mathrm{PO}(n,1) = \mathbf{G}(\mathbb{R})$ commensurable with $\mathbf{G}(\mathcal{O}_k)$ for some admissible k-group $\mathbf{G}$ is said to be an *arithmetic subgroup.*

Example 2.6. Let us consider be the quadratic form $f = -\sqrt{2}x_0^2 + x_1^2 + \cdots + x_n^2$. It is defined over $k = \mathbb{Q}(\sqrt{2})$, and so is the algebraic group $\mathbf{G}$ obtained as the image of $\mathrm{SO}(f)$ under the (algebraic) adjoint representation. The group $\mathbf{G}$ is admissible, and thus determines a commensurability class of arithmetic subgroups in $\mathrm{PO}(n,1)$.

By the construction, an arithmetic subgroup of $\mathrm{PO}(n,1)$ is a lattice, and we simply speak of an *arithmetic lattice.* Often one allows a slightly larger class of admissible groups in the definition (see [6, Ch. 1 Sect. 7.4]). This permits more flexibility in the construction, but does not enlarge the class of arithmetic subgroups. The advantage with our definition of admissibility is that $\mathbf{G}$ has necessarily trivial center, from which we have the following (see [2, Prop. 1.2]).

Proposition 2.7. *Let $\Gamma \subset \mathbf{G}(\mathbb{R}) = \mathrm{PO}(n,1)$ be arithmetic, commensurable with $\mathbf{G}(\mathcal{O}_k)$ for $\mathbf{G}$ admissible. Then $\Gamma \subset \mathbf{G}(k)$.*

3 Quasi-arithmetic lattices

In his paper [11] Vinberg obtained a general criterion for checking if a given hyperbolic reflection group is arithmetic or not. To express this criterion he introduced the following notion (in a slightly different – but equivalent – way).

Definition 3.1. We say that a lattice $\Gamma \subset \mathrm{PO}(n,1)$ is *quasi-arithmetic* if $\Gamma \subset \mathbf{G}(k)$ for some admissible k-group $\mathbf{G}$ (where we identify $\mathbf{G}(\mathbb{R}) = \mathrm{PO}(n,1)$).

By Proposition 2.7, any arithmetic lattice is quasi-arithmetic. We say that $\Gamma \subset \mathrm{PO}(n,1)$ is *properly* quasi-arithmetic if it is quasi-arithmetic but not arithmetic. The existence of properly quasi-arithmetic lattices is a priori not clear. An example for $n = 4$ was already given in [11]. Recently, the construction of Belolipetksy and Thomson [1] allowed to show their existence in any dimension, as was proved by Thomson in [10].

Theorem 3.2 (Thomson). *For any $n \geq 2$ there exist infinitely many commensurability classes of properly quasi-arithmetic lattices in* $\mathrm{PO}(n,1)$.

The main idea for the construction that produces the lattices in Theorem 3.2 is to start with an *arithmetic* hyperbolic n-manifold and to cut it along two nonintersecting totally geodesic submanifolds of codimension 1. The piece obtained is a hyperbolic manifold with boundary, and gluing together two copies of it along their (isometric) boundaries one obtains a hyperbolic manifold without boundary. We call it an *inbreeding* manifold. An argument about the lengths of geodesics shows that very often this manifold cannot be arithmetic. The inbreeding construction is a variation of the classical *interbreeding* construction of Gromov and Piateski-Shapiro [4], which produces nonarithmetic hyperbolic manifolds by gluing together pieces of noncommensurable arithmetic manifolds.

In [10] Thomson showed that the inbreeding construction always gives quasi-arithmetic lattices, whereas the interbreeding construction does not. This already motivates the notion of quasi-arithmeticity, as a useful tool for the classification of hyperbolic manifolds.

4 Volumes of quasi-arithmetic lattices

Before the proof of Theorem 3.2 in 2015, there is no mention in the literature of the notion of quasi-arithmetic lattices that is not directly related to Vinberg's arithmeticity criterion. Thus it appears that this quite natural generalization of the notion of arithmetic lattices has been understudied.

The author could recently prove the following result about quasi-arithmetic lattices; see [3]. The idea of the proof (which we will not discuss here) is summarized in Sect. 1.5 in *loc. cit.*

Theorem 4.1. *Let $\mathbf{G}$ be an algebraic k-group that is admissible for* $\mathrm{PO}(n,1)$. *Then for any quasi-arithmetic lattice $\Gamma \subset \mathbf{G}(k)$ we have that* $\mathrm{vol}(\Gamma\backslash\mathbf{H}^n)$ *is commensurable with* $\mathrm{vol}(\mathbf{G}(\mathcal{O}_k)\backslash\mathbf{H}^n)$.

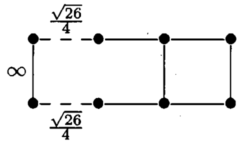
Remark 4.2. The result is known for n even: it follows from the (generalized) Gauss-Bonnet formula that all volumes are rationally proportional (for n fixed).

Remark 4.3. If Γ is arithmetic the result is also clear for any n: by definition Γ must be commensurable to $\mathbf{G}(\mathcal{O}_k)$, and in particular their covolumes are commensurable.

Remark 4.4. The result is known in dimension 3, and follows from the Bloch invariant theory (see [5]).

For $n > 3$ odd and Γ properly quasi-arithmetic Theorem 4.1 provides new information about the volume of Γ. An important aspect to bear in mind is that the volume of arithmetic lattices is fairly well understood, thank to the methods developed (notably) by Siegel, Weil, Tamagawa, Ono [7], and Prasad (see [8]). In particular, for any $\mathbf{G}$ admissible the value of $\mathbf{G}(\mathcal{O}_k)\backslash\mathbf{H}^n$ is known up to a rational. Thus in principle the volume of any quasi-arithmetic lattice $\Gamma \subset \mathbf{G}(k)$ is known up to a (nonexplicit!) rational, as long as enough information on $\mathbf{G}$ is given. We illustrate this in the next example (see [3, Sect. 2] for details).

Example 4.5. Denote by $\Delta_5 \subset \mathrm{PO}(5,1)$ the following hyperbolic Coxeter group, which was recently discovered by Roberts (see [9]).



(4.1)

Using Vinberg's criterion [11] we know that Δ_5 is quasi-arithmetic, sitting inside $\mathbf{G}(\mathbb{Q})$ for some admissible $\mathbb{Q}$-group. Moreover, the structure of $\mathbf{G}$ can be obtained from the Gram matrix of Δ_5. From Prasad's volume formula [8] we have that the covolume of $\mathbf{G}(\mathcal{O}_k)$ is commensurable to $13^{5/2} \cdot \frac{\zeta_\ell(3)}{\zeta(3)}$, where ζ_ℓ is the Dedekind zeta function attached to the field $\ell = \mathbb{Q}(\sqrt{-13})$. Thus, by theorem 4.1 the same should hold for Δ_5. Using a geometric integration on the fundamental domain of Δ_5 (described as a noncompact Coxeter polytope), Steve Tschantz computed the following numerical approximation that indicates (almost certainly!) that the covolume of Δ_5 must equal the following value

$$\frac{1}{23040} \cdot 13^{5/2} \cdot \frac{\zeta_\ell(3)}{\zeta(3)}. \tag{4.2}$$

It appears that Theorem 4.1 has also some theoretical importance. First, it permits to describe the nature of the volumes of all hyperbolic manifolds that are obtained by either the inbreeding or the interbreeding construction; see [3, Sect. 1.4]. More surprisingly, the methods developed to prove Theorem 4.1 has also consequences for the volume distribution of arithmetic lattices. For instance, the following result (see Corollary 1.9 in *loc. cit.*).

Proposition 4.6. *Let $\Gamma \subset \mathrm{PO}(n,1)$ be an arithmetic subgroup. There exists a number $c > 0$ for which the covolume of any subgroup commensurable to Γ is an integral multiple of c.*

It seems that for $n > 3$ this result is new (also for n even).

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